Lecture 10.

Recall the definition of the Cantor set (generalized). Given $\{\alpha_k\}_{k=1}^{\infty}$ w/ $\alpha_k \in (0,1)$, construct $\mathcal{C}_k^{\alpha}$ by removing the open "middle α_k-th interval" of each interval in $\mathcal{C}_{k-1}^{\alpha}$. The Cantor set $\mathcal{C}^{\alpha} = \bigcap_{k=1}^{\infty} \mathcal{C}_k^{\alpha}$. The construction and cont. from above yields

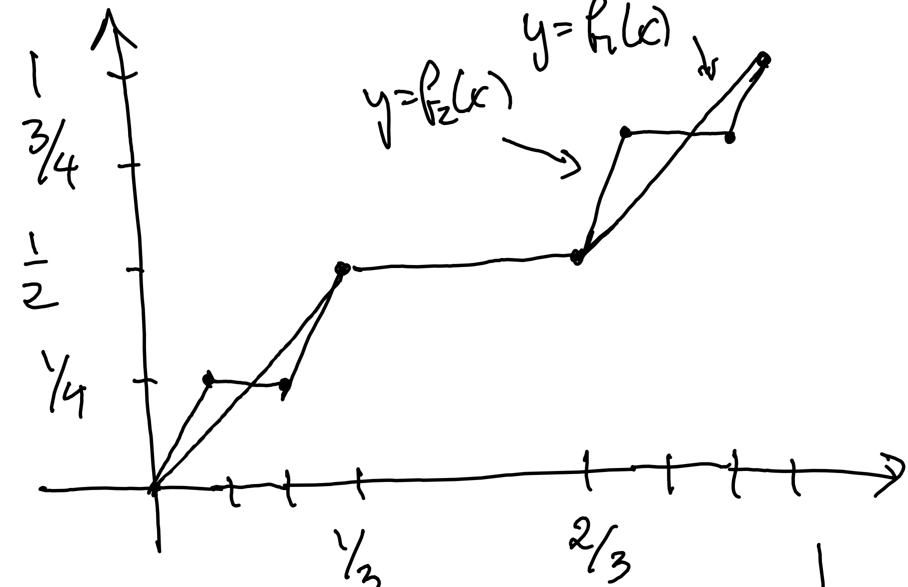
$$m(\mathcal{C}^{\alpha}) = \prod_{k=1}^{\infty}(1-\alpha_k) = \begin{cases} e^{\sum \log(1-\alpha_k)} & \text{if } \sum \alpha_k < \infty. \\ 0 & \text{if } \sum \alpha_k = \infty. \end{cases}$$

<u>Prop 1.</u> The Cantor set $\mathcal{C}^{\alpha}$ is compact and contains no interval ($\Rightarrow$ totally disconnected and nowhere dense). Moreover, $\mathcal{C}^{\alpha}$ has no isolated points.

Pf. $\mathcal{C}$ is closed and bdd by construction $\Rightarrow$ $\mathcal{C}$ is compact. Since $\mathcal{C} = \bigcap_{k=1}^{\infty} \mathcal{C}_k$ and each $\mathcal{C}_k$ consists of 2^k intervals of length $\frac{1}{3^k}$ (or in general $\frac{(1-\alpha_1)\cdots(1-\alpha_k)}{2^k} \to 0$) it is clear that no interval I of positive length can be contained in $\mathcal{C}_k$ for k large enough. To see that no $x \in \mathcal{C}$ is isolated, note that $x \in \mathcal{C}_k$, $\forall k$, and hence x belongs to some interval of length $\frac{1}{3^k}$, $\forall k$. It is also easy to see that the countable collection of points obtained as endpoints of the intervals in $\mathcal{C}_k$, $\forall k$, belongs to $\mathcal{C}$. The conclusion that x is isolated follows $\square$

The Cantor-Lebesgue Function.

Consider the piecewise linear functions



Each f_k is constant on $C_k^c \cap [0,1]$ and linear, increasing from $(j-1)\frac{1}{2^k}$ to $j \cdot \frac{1}{2^k}$ on the j:th interval (of length $\frac{1}{3^k}$) in C_k.

Easily verified properties:

- f_n are cont. and map $[0,1]$ onto $[0,1]$
- f_n is constant on $C_n^c \cap [0,1]$.
- For any $k \le l$,

$$\sup_{x\in[0,1]} |f_k(x) - f_l(x)| \le \frac{1}{2^k}$$

$\Rightarrow$ $\{f_n\}$ uniformly Cauchy seq. in $C([0,1])$.

$\Rightarrow$ $\exists\, f \in C([0,1])$ s.t. $f_n \to f$ uniformly.

- f is increasing and maps $[0,1]$ onto $[0,1]$.

- $\mathcal{C}^c \cap [0,1]$ is a countable union of open intervals and f_k is constant on each one for k large enough.

  $\Rightarrow$ f is constant on $\mathcal{C}^c \cap [0,1]$

  $\Rightarrow$ $f|_{\mathcal{C}}$ is a map $\mathcal{C}$ onto $[0,1]$.

$\Rightarrow$

<u>Thm 1.</u> There is a cont., increasing function $f : [0,1] \to [0,1]$ s.t.

(i) f is constant on $\mathcal{C}^c \cap [0,1]$

(ii) $f|_{\mathcal{C}}$ is surjective onto $[0,1]$.

<u>Rem.</u> (ii) $\Rightarrow$ card $(\mathcal{C})$ = card$([0,1])$.

<u>Cor 1</u>. The Lebesgue-Cantor function is increasing, cont., surjective function $f : [0,1] \to [0,1]$ and there is a dense open set $U := \mathcal{C}^c \cap [0,1]$ with $m(U^c \cap [0,1]) = 1$ and f is cont. differentiable on U with $f'(x) = 0$, $\forall x \in U$.